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Ocean-Wave Record Analysis -
Ordinate Distribution and Wave Heights

By

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Ocean-Wave Record Analysis - Ordinate Distribution and Wave Heights

ABSTRACT

The methods of analysing ocean wave records into ordinates and into waves are compared. It is suggested that suitable wave spectra will permit many of the numerical results of one procedure to be obtained from the other. The Rayleigh distribution is introduced as that of the envelope of a Gaussian random process, which approximates the peaks and troughs of the process curve in the case of narrow-band spectra. This distribution is shown to resemble wave-height distributions closely in regard to shape, dependence on a single parameter, and values of various parameter ratios. The single parameter may be interpreted as the dispersion (r.m.s. deviation) of the ordinate distribution, or the total power in the wave spectrum.

The ordinates of twenty-one time histories analyzed on the Wave-Ordinate Distribution Analyzer are found to follow the Gaussian distribution closely enough to permit ready graphical estimation of the ordinate dispersion. The mean wave height is found to be closely proportional to the latter quantity, the proportionality factor being about 15% less than that for the mean envelope height. The scatter around the straight line of proportionality is small enough to indicate that measurement of mean wave height may for many purposes be replaced by measurement of wave-ordinate dispersion, which is more suited to analog computation.

The present data are used to illustrate simplified wave-ordinate distribution analyses yielding reasonably accurate estimates of ordinate dispersion and, hence, mean wave height. There is suggested an ocean-wave measurement program based on these results designed to provide a nearly-continuous flow of immediate wave-height information over a recording period extending over several seasons.

1. Introduction

Visual observation of the surface of the sea has in many cases led to its analysis into a series of crests and troughs. For a mildly irregular sea recorded at a point, such features appear on a time history of the surface or of the subsurface pressure as well-defined prominences and depressions on a continuous curve. The more regular the fluctuation, the more nearly is this type of description adequate, and in the limiting case of perfect regularity, knowledge of the height of any one wave and the period between any two successive wave crests is sufficient to characterize the resulting sinusoidal wave. The more irregular the time history, on the other hand, the less complete will be the information contained in the results of an enumeration of "wave heights" and "wave periods". In fact, it has been found that for typical records of ocean swell the entire statistical distribution of estimated surface wave heights over a twenty-minute interval yields essentially only one number for characterizing the wave record. In view of the discussion below, this may be taken as evidence that such wave systems have spectra with frequency bandwidths which are narrow relative to the central frequency of the band. It is reasonable, of course, to assume that the spectra of wave records having similar geophysical origins have important common features characterizing their shape.

Such theories suggest that certain statistical information about wave heights and periods may eventually contribute substantially to an adequate description of the state of the sea.

However, until more is learned about the form of typical ocean-wave spectra, it is natural to direct attention to the continuous aspects of the wave-record curve, rather than to its discrete wave-by-wave characteristics. The concept of a Fourier spectrum(5)* then becomes available, together with the results of other linear operations performable on the ordinates of the time-history function, some of which may be conveniently arrived at by analog computation.

The purpose of the present report is to present certain experimental results which relate the ordinate-wise and wave-wise approaches to the analysis of ocean-wave records for general wave-height. These relations, which may in part be deduced from certain assumptions of randomness and spectral narrowness, appear to be of interest both as explanations of previously-known wave-height phenomena and as possible means for the indirect measurement of wave heights.

2. Narrow-spectrum Theory of Wave Heights

If an observed ocean-wave time history $f(t)$, measured about its mean value, be regarded as a sample from a stationary ergodic random process, the concept of an associated envelope process may be defined. This has been done(14) by introducing the conjugate process $f^*(t)$, which may be thought of as being obtained from $f(t)$ by shifting each of the angles in its spectrum through $\pi/2$ radians. The envelope, defined by $R(t) = \left\{[f(t)]^2 + [f^*(t)]^2\right\}^{\frac{1}{2}}$, will be a random process, the ordinates on the graph of which will follow some probability distribution whose nature depends upon the distribution of both the ordinates and the spectral energy of the original process $f(t)$. It may be noted that $R(t) \geqslant f(t)$. For the limiting case in which $f(t) = a \sin w\, t$, we have $f^*(t) = a \cos w\, t$, and the envelope has the constant value given by $R(t) = |a|$. For sufficiently narrow-band spectra, $R(t)$ and $[R(t)]^2$ may be obtained in principle as the electronically-filtered outputs of ideal linear or square-law detectors.

The envelope curve as here defined may be shown(14) to possess the property intuitively expected of it - namely, that it pass near the extreme points on the original curve - provided the relative bandwidth of the spectrum of the latter is small. It is thus not unreasonable, in view of the appearance of ocean swell records, to expect the distribution of the wave heights on the original curve to show some agreement with the distribution of the ordinates of its envelope.

*See list of references at end of report.

Certain assumptions on the nature of the wave-record spectrum have been shown[8],[21] to lead to a definite type of ordinate distribution for the envelope curve R(t). The assumption chosen here, which is in part, at least, directly verifiable, is that expressed by the condition that f(t) be a Gaussian random process[2],[10],[11]. This assumption has the special consequence that the ordinates of the original curve have a one-dimensional Gaussian distribution. That is, for any y and for any $dy > o$, the probability that $y < f(t) \leq y + dy$ is given by $1/(2\pi\sigma_o^2)^{\frac{1}{2}} \exp\left[-y^2/(2\sigma_o^2)\right] dy$, where $\sigma_o^2 = \overline{[f(t)]^2}$, the bar denoting the operation of averaging. A further consequence of the Gaussian hypothesis for a stationary ergodic random process is that, in a statistical sense, f(t) and R(t) are described completely by the unnormalized correlation function[14] $\gamma(\tau) = \overline{f(t)f(t+\tau)}$. It may be noted that $\gamma(o) = \sigma_o^2$, a fact which might be utilized for indirect measurement of the correlation function.

Under these assumptions it may be shown that the ordinates of the envelope follow a Rayleigh distribution. That is, for any $r \geq o$ and any $dr > o$, the probability that $r < R(t) \leq r + dr$ is just $(1/\sigma_o^2) r \exp\left[-r^2/(2\sigma_o^2)\right] dr$, while the probability that $R(t) > r$ is given by

$$\exp\left[-r^2/(2\sigma_o^2)\right] \qquad \ldots (1)$$

It will be seen that both Rayleigh and Gaussian distributions depend on the single parameter σ_o^2. The positive quantity σ_o, the standard deviation, is a measure of the dispersion of the ordinates to the wave-record curve taken about their mean value (assumed here to be zero). Physically, for purposes of measurement it may be interpreted as the root-mean-square value of the time-history function measured, again, from its mean. Its square, σ_o^2, may be interpreted as the average total power (energy per unit time) for the wave spectrum of long-crested linear wave theory.

The parameter of these distributions is directly related to the mean value and dispersion of the envelope distribution. It may be shown that the latter parameters are respectively

$$\mu_R = \sigma_o\,(\pi/2)^{\frac{1}{2}} = 1.253\,\sigma_o, \qquad \ldots (2)$$

and

$$\sigma_R = \sigma_o\left[(4-\pi)/2\right]^{\frac{1}{2}} = 0.655\,\sigma_o, \qquad \ldots (3)$$

their ratio, called the relative dispersion, being

$$\sigma_R/\mu_R = \left[(4-\pi)/\pi\right]^{\frac{1}{2}} = 0.523. \qquad \ldots (4)$$

Further, a measure related to distribution asymmetry, or skewness, previously applied[13] to wave-height distributions, is found to have the value

$\alpha_3 = +0.63$ for the Rayleigh distribution. Finally, the mean, $R_{1/3}$, of the highest one-third of the envelope ordinates is given by

$$R_{1/3} = \sigma_0 \left\{ (\ln 9)^{\frac{1}{2}} + 3 \left[(2\pi)^{\frac{1}{2}} - \int_{-\infty}^{(\ln 9)^{\frac{1}{2}}} \exp(-t^2/2)\, dt \right] \right\}$$

$$= 2.002\,\sigma_0 = \mu_R + 1.143\,\sigma_R = 1.597\,\mu_R. \quad \ldots (5)$$

For an illustration of the way in which the Rayleigh distribution assigns different probabilities to different vertical intervals on the wave-record chart, one may refer to Figure 1, where the straight lines represent Rayleigh distributions. The horizontal scale has been laid off in accordance with (1), so that on a straight line through the origin a point with ordinate r has as abscissa the probability that $R(t) \leq r$, i.e. the percent of the time the envelope curve is no higher than r. The mean envelope height μ_R is then the intercept cut off by the vertical line erected at the 54.4% level (denoted by the caption "Mean" on Figure 1.) The standard deviation of the envelope is given by the difference between the intercepts of the line in question by the two lines labelled "Mean + Dispersion" (erected at the 83.8% level) and "Mean". The standard deviation of the ordinates on the original wave record having the given envelope distribution would be $\sigma_0 = \mu_R (2/\pi)^{\frac{1}{2}}$. The lines on the figure correspond to various given values of σ_0, the units being the same as those for the envelope ordinate $R(t)$. It is seen, for instance, that when $\sigma_0 = 2.0$ units we have 95.6% of the envelope ordinates below 10.0 units and 98.9% below 12.0 units, so that 3.3% of the envelope ordinates are between 10.0 and 12.0 units.

The intercept cut off by the line erected at 86.5%, labelled "$R_{1/3}$" in Figure 1, is equal to the mean of the highest one-third of the envelope ordinates.

3. Empirical Results on Estimated Surface Wave Heights

Previous work[12],[13] has led to results on the distributions of wave heights for ocean swell in shoaling water. These heights were measured from trough to crest on twenty-five twenty-minute subsurface pressure records and then modified in an attempt to estimate surface wave heights. A general description of these wave records appears in Table I. The results obtained which are of present interest are recounted below.

The dashed curve in Figure 2 represents the mathematical distribution type earlier found to provide a good fit to the estimated ocean surface wave-height distributions. This model distribution is the member of the family of Gamma-type distributions with the skewness coefficient $\alpha_3 = +0.8$. It may be compared with the dashed straight line corresponding to the Rayleigh distribution. There has also been drawn on this figure a full-line curve representing the Gamma-type distribution having $\alpha_3 = +0.6$, which is the median value for the twenty-one linear wave records examined in the present study, and the typical value found earlier[3] for wind-generated waves. All three distributions were chosen to have the same mean and standard deviation, so that their shapes, or types, could be compared. It is seen that the previous data show little evidence for rejecting the hypothesis that estimated surface swell heights follow a Rayleigh-type distribution.

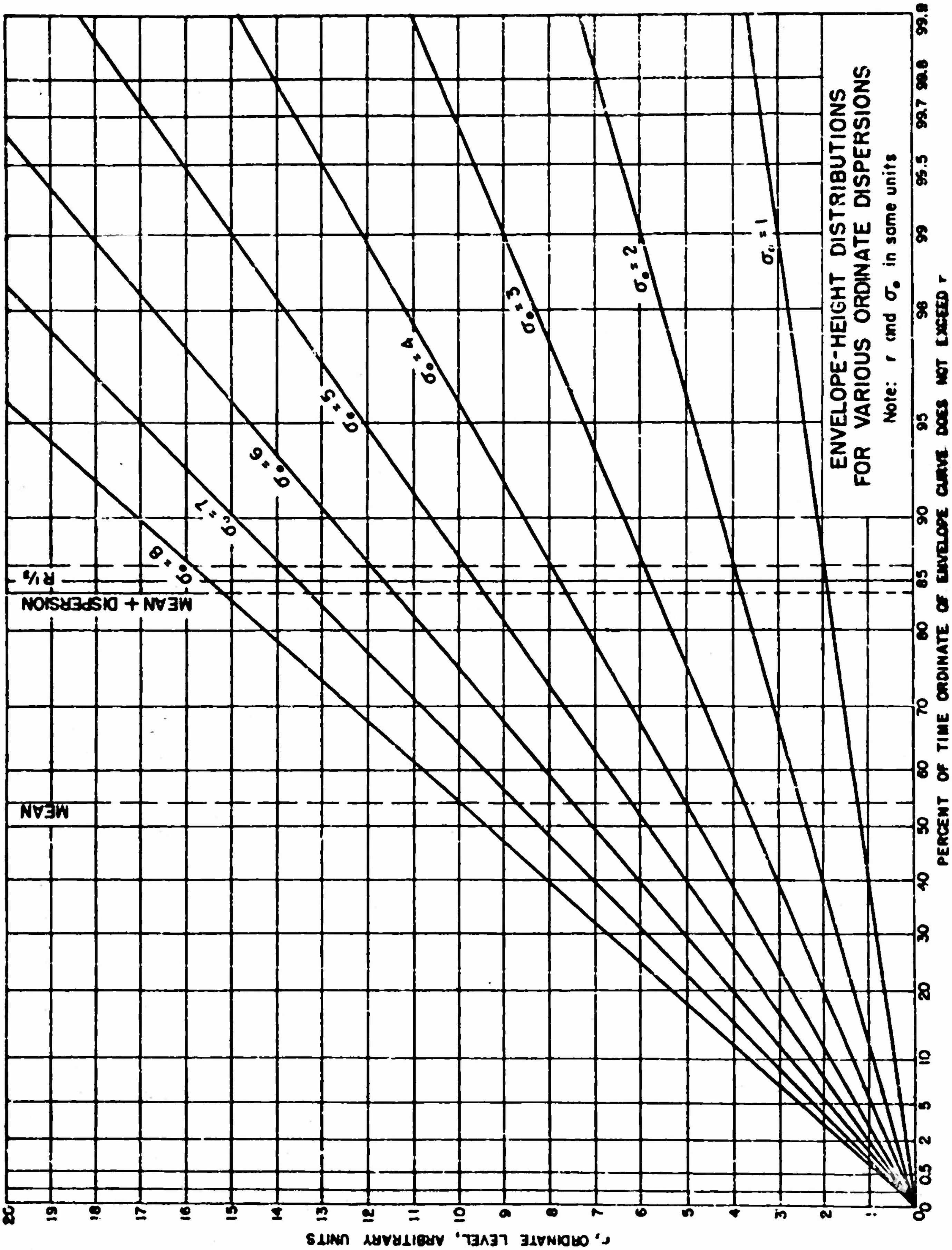

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FIGURE 1

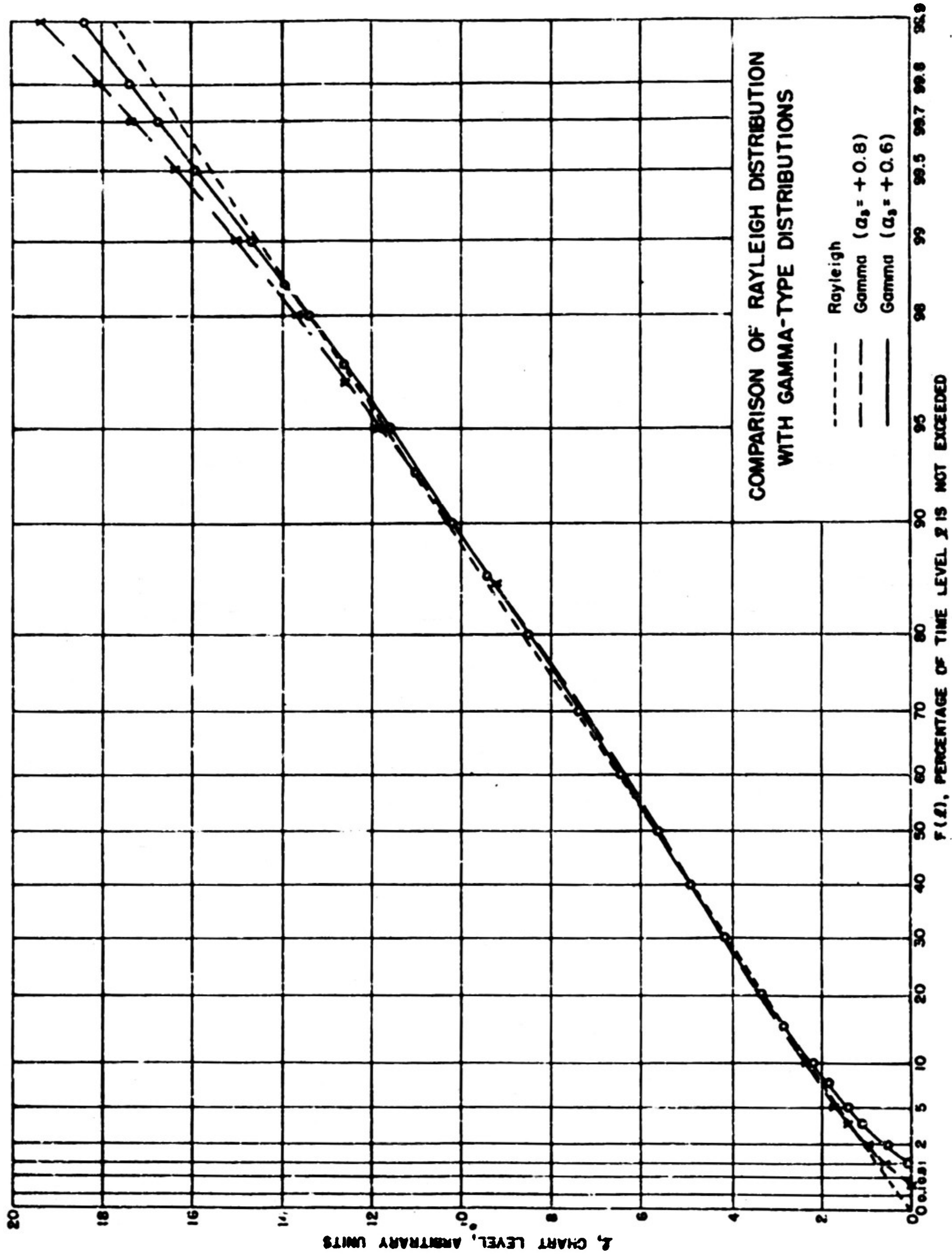

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FIGURE 2

Table I Descriptive data for Mark III pressure records

Record Identification	Location	Depth ft.	Date	Time h m h m	Mean Wave Height μ_H Chart div.	Ordinate Dispersion σ_0 Chart div.
E	Pt. Sur, Calif.	64	Sept. 23, 1948	09 15 - 09 35	9.92	4.33
F		64	Sept. 23, 1948	15 15 - 15 35	9.13	4.22
G		64	Sept. 23, 1948	21 19 - 21 39	10.83	5.13
H		64	Oct. 3, 1948	22 20 - 22 40	9.04	4.10
I		64	Oct. 4, 1948	04 18 - 04 38	11.59	5.43
J		64	Oct. 5, 1948	10 18 - 10 38	4.16	2.00
K		64	Oct. 11, 1948	21 56 - 22 16	18.77	8.57
L		64	Oct. 12, 1948	03 56 - 04 16	13.58	6.66
M		64	Oct. 12, 1948	09 56 - 10 16	12.34	5.80
N		68	Nov. 28, 1947	23 07 - 23 27	18.42	8.28
O		64	Nov. 4, 1948	16 48 - 17 11	22.44	10.47
P		64	Nov. 4, 1948	22 44 - 23 11	21.43	9.52
Q		68	June 2, 1949	05 33 - 05 53	15.46	6.96
R		68	June 2, 1949	17 40 - 18 00	13.21	5.91
S		68	June 2, 1949	23 42 - 24 02	15.10	6.73
T	Heceta Head Oregon	53	Sept. 8, 1947	16 04 - 16 24	4.36	1.91
U		53	Dec. 30, 1947	16 16 - 16 40	1.78	1.03
V	Guam, M.I.	60	Feb. 17, 1949	05 17 - 05 38	30.68	14.13
W		60	Feb. 18, 1949	03 23 - 03 43	22.15	10.64
X		60	Feb. 18, 1949	20 37 - 20 58	17.04	7.22
Y	Pt. Arguello, Calif.	83	July 22, 1949	22 00 - 22 22	7.16	3.77

The same earlier investigation established the dependence of the wave-height distribution on essentially just one parameter. This could be taken as either the mean or the dispersion of the wave-height distribution, since it was found that these two parameters were quite regularly in the average ratio $\sigma_H/\mu_H = +0.53$ - in close agreement with the relative dispersion of the Rayleigh distribution.

This agreement between estimated wave heights and envelope, both in distribution shape and parameter ratios, may be interpreted as evidence that the wave records studied were characterized by relatively narrow-band spectra. The regularities in wave-height distributions observed earlier were used to explain the regular values of certain nearly-constant ratios between various wave-height statistics observed by other investigators(17),(19),(20),(22). It is now seen that a more fundamental explanation of both types of regularity is provided by the narrow-band envelope theory combined with the Gaussian hypothesis for the wave-record ordinates.

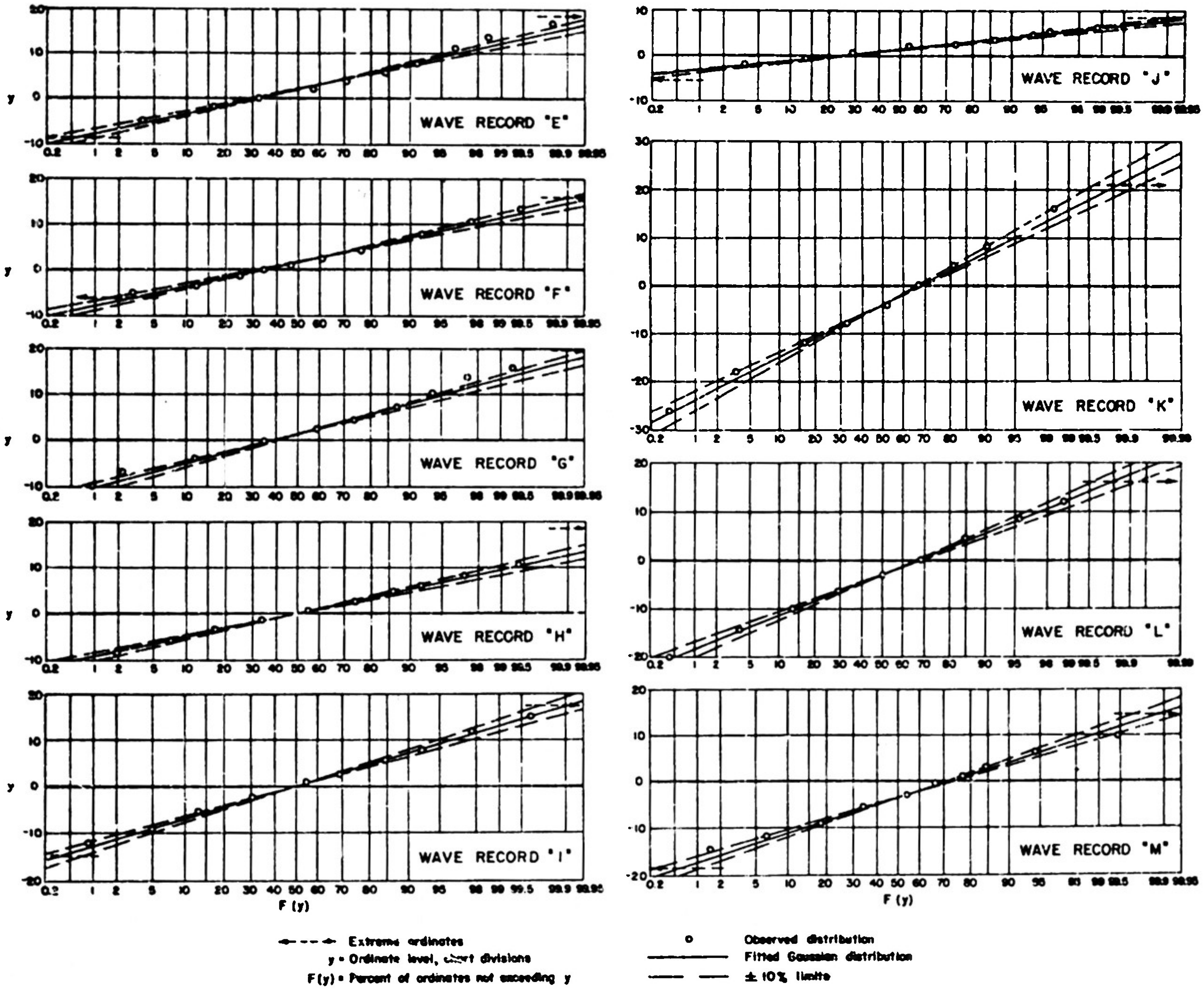
4. Pressure Wave Ordinate Results

More direct verification of the theory presented above has been possible. Information about the distribution of ordinates on the pressure wave-record curve was obtained in January 1952. This distribution was found to be closely related to the distribution of pressure wave heights.

For this investigation, the Wave-Ordinate Distribution Analyzer proposed by and constructed under the supervision of F.E. Snodgrass was used. This electronic equipment, described elsewhere(9) in detail, receives as input the varying voltage output of a pressure meter, or of a tape or other recording of this voltage signal. For the analysis described here, the twenty-minute pressure records previously(12),(13) analyzed in a wave-wise manner were retraced by a human curve-follower, producing a replica of the original varying-voltage signal, containing, however, a relatively small amount of added "noise" due to errors in tracing. The analyzer performed, in effect, a sampling of the ordinates to the voltage curve every 0.1 second, and by means of ten recording counters set at arbitrarily-chosen voltage levels, accumulated the total number of sampled ordinates below each of the given levels.

In practice, the counter voltage levels are spaced at nearly equal intervals covering the entire range of ordinates, so that when each counter reading is divided by the total number (approximately 12,000) of ordinates sampled, a definitive set of points on the cumulative distribution curve for the ordinates is obtained.

The empirical ordinate distribution functions for the twenty-one records made by the Mark III Pressure Meter (7) are plotted in Figure 3. In each case, the ordinate is an arbitrary differential pressure level measured in chart divisions and the abscissa is the percent of the sampled pressure-record ordinates which do not exceed that level. It will be seen that the plotted points appear in each case to lie on an approximately straight line. Since the abscissa scale in the figure has been taken to correspond to a Gaussian probability distribution (6), the observed results indicate a



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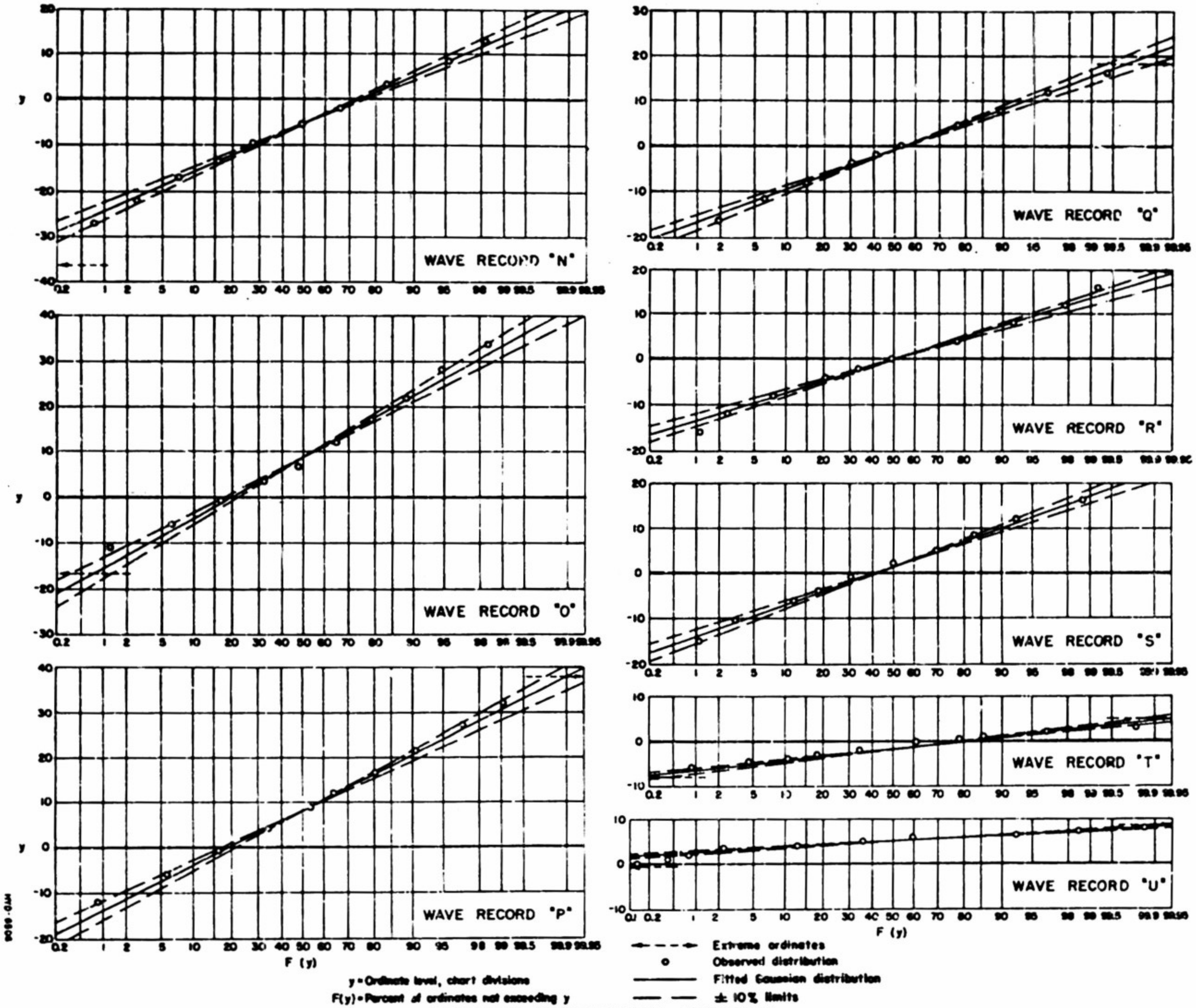
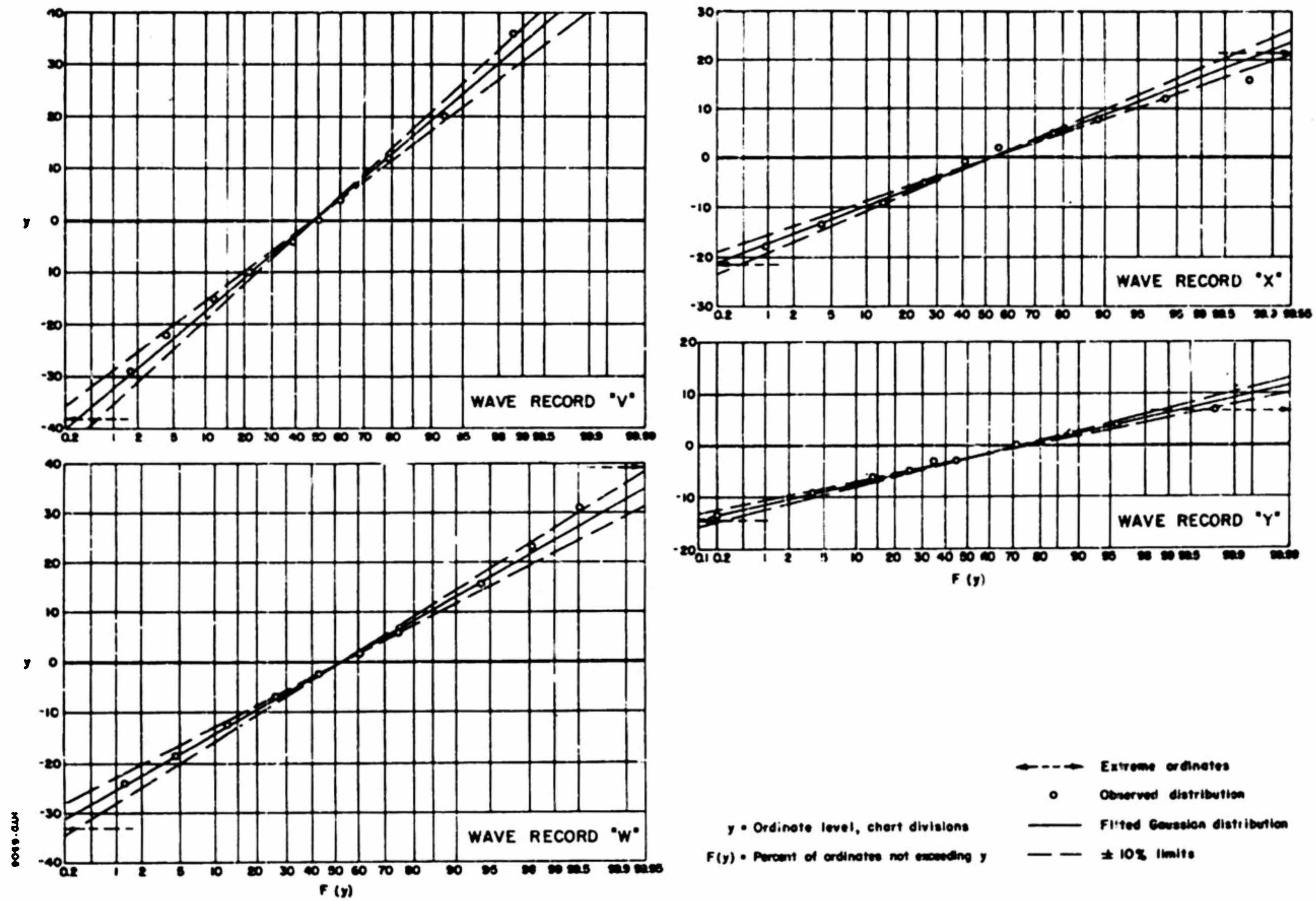
ORDINATE DISTRIBUTION FUNCTION

FIGURE 3a



ORDINATE DISTRIBUTION FUNCTION

FIGURE 3b



ORDINATE DISTRIBUTION FUNCTION

FIGURE 3c

Gaussian distribution of pressure-record ordinates. Similar results for wave records have been reported by J.E. Meade using the distribution density analyzer at the Naval Research Laboratory, and in the literature.(10),(11),(16)

Even with the Gaussian random process model described above, the plotted points in Figure 3 would not be expected to lie on straight lines as long as the wave records were of finite length. A measure of the actual sampling fluctuation to be expected for the abscissa of any one point on these plots may be computed from the wave spectrum or, equivalently, from its correlation function. The amount of such fluctuation also depends upon the temporal spacing between consecutive ordinates sampled from the wave record. When the correlation function (or the spectrum) for the wave record is also known, it is possible to assess the likelihood that the sampled ordinates come from a Gaussian distribution of ordinates. It may be noted that the familiar "chi-square" test often used for this purpose is based upon independent observed random variables, i.e. in the present instance upon ordinates from a wave record possessing a completely flat and broad spectrum. Such a situation would not necessarily be expected to hold, e.g. for a sampling of 200-300 ordinates from a 20-30-minute pressure record for a typical correlation function indicating a moderately narrow spectrum centered around 10-15 seconds, even if the sampling is done in some random manner(11).

The result of the analysis of Wave Record "B", made with the Mark V Thermopile Wave Meter(4), is shown in Figure 4. The non-Gaussian character of the ordinate distribution for this wave record, which has been noted by Birkhoff and Kotik(2), is shared by Wave Records "A", "C", and "D", which are the other three Mark V records among the original twenty-five. Owing to the non-linear response of the Mark V instrument, the results are readily interpreted as distortion of the Gaussian record which would have been obtained with the linear Mark III instrument. Of the four non-linear time histories, only Wave Record "B" was analyzed by the Wave Ordinate Distribution Analyzer. For the other three, visual inspection alone was sufficient to show considerable positive skewness in the ordinate distributions. This characteristic was also noted by Dr. Philip Rudnick, who in 1951 analyzed Wave Record "B" and several other records on the distribution analyzer at the U.S. Naval Electronics Laboratory(15).

In regard to the slight non-linearity of the Mark III data, no interpretation of the curvature to be seen in some of the plots in Figure 3 has been made. Such curvature, if concave toward the crests, i.e. upward, and persisting from record to record, would be consistent with an expected non-linearity(2),(10), in the wave records. However, on the records analyzed(12) it is not generally known which is the direction toward the crest and which is that toward the trough. The magnitude of the depth, varying between 50 and 85 feet, at which the pressure records were made is probably responsible for there being no discernible relation between the slight degree of curvature in the plots in Figure 3 and either mean wave height or depth of recording.

The twenty-one plots in Figure 3 not only provide evidence for the existence of underlying Gaussian ordinate distributions, but the individual plots may

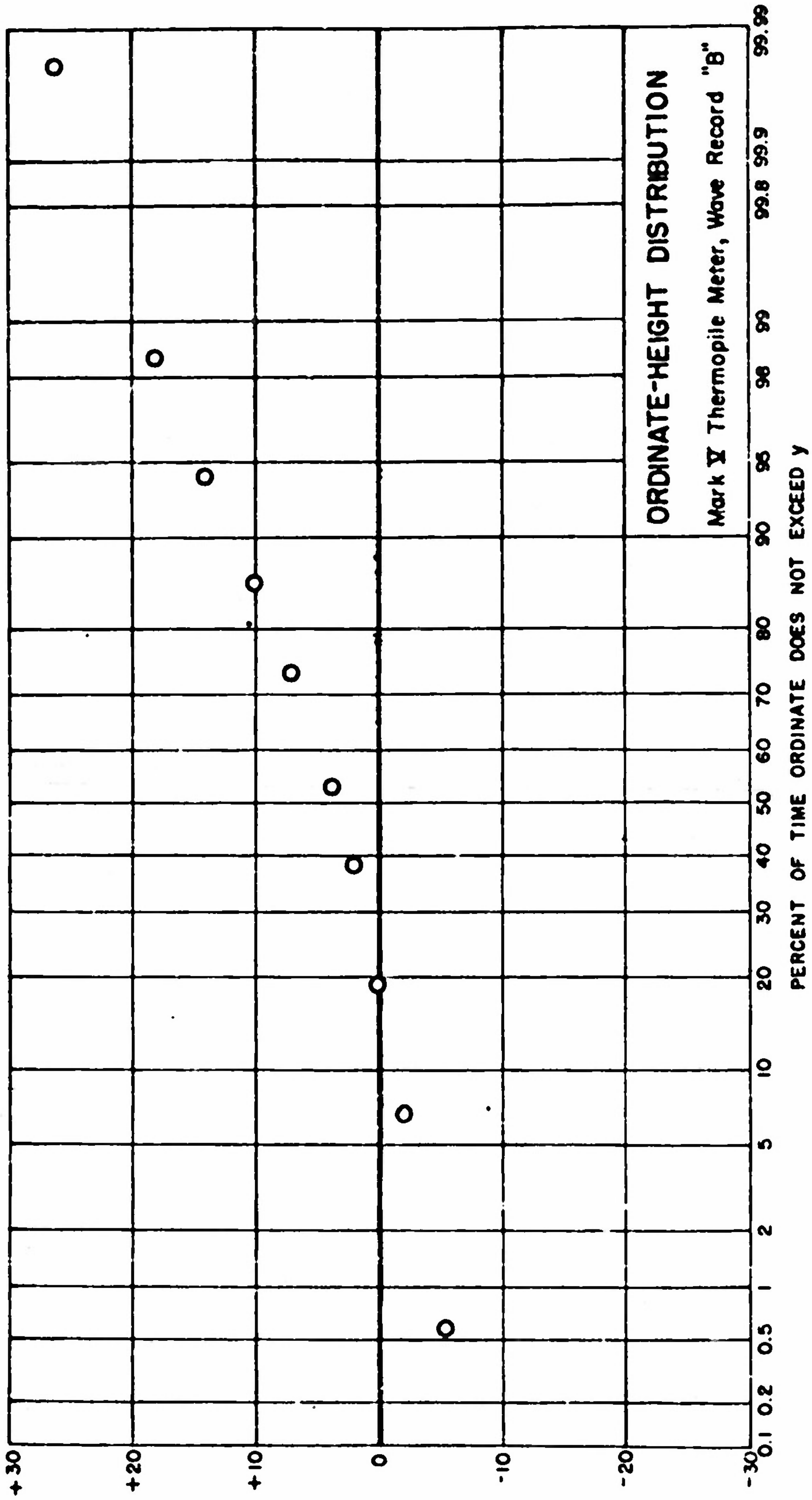


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FIGURE 4

also be used to estimate the parameters of these distributions. Various general methods of estimating the ordinate dispersion σ_o and the ordinate mean μ_o were employed, in order to investigate both the validity of the envelope theory and the accuracy obtainable in rapid graphical estimation of these parameters. Only two of these methods, referred to as the methods of "direct computation" and "graphical estimation", are discussed here and in Section 6. Chief interest in the present investigation was in the relation between the ordinate dispersion and the distribution of wave heights. For many purposes an estimate of the mean ordinate μ_o may be required, as when measurements are to be made from "still-water level", as assumed in the theory.

The ordinate-distribution parameters for each wave record were computed directly from the statistical frequency distribution composed of finite-width classes defined by the counter levels and the counter readings. This computation, to which a correction for the finite average class-width was applied, is regarded, for purposes of comparison with the theory, as yielding the most reliable estimates of σ_o and μ_o, which are taken as standards for comparison with wave-height statistics and other subsequent estimates of these distribution parameters. The directly-computed values of σ_o appear in Table I. Because the set of counter levels used in analyzing the data rarely included the entire range of ordinates present on the wave record, extension of the classes provided by the counter levels was made. This was done, so as to give ten classes, by assuming that each end-class required to encompass one-hundred percent of the ordinates would have the same width as the class adjacent to it.

The full lines appearing on Figure 3 represent Gaussian distributions, their intercept and slope being in each case the directly-computed mean and standard deviation. The fit of each of the lines to the pressure time-history ordinate data is seen to be reasonably good.

5. Pressure Ordinates and Wave Heights

As a check on the application of the narrow-band envelope theory to pressure records of ocean swell, the relation between the observed ordinate dispersion and mean trough-to-crest wave height was examined. For this purpose, a wave height on the twenty-one pressure records was counted and measured between each minimum (trough) and the immediately following maximum (crest). The mean value of these maximum-minimum pressure differences is referred to as the mean wave height μ_H. The computed values of this quantity are shown in Table I. The relation between μ_H and the computed value of σ_o is shown in Figure 5, together with the theoretical straight line for the mean $2\mu_R$ of the double-envelope, whose equation, obtained by doubling both members of (2), is $2\mu_R = 2\sigma_o(\pi/2)^{\frac{1}{2}}$. Agreement with this line would be expected only, of course, if measurements were made on the true envelope of an infinite-length wave record satisfying the Gaussian process hypothesis. A straight line fitted to the data is seen to be approximately 15% below the envelope line. The slope of the least-squares line through the origin shown in Figure 5 is, in fact, 0.850 times the slope of the envelope line. This systematic deviation of the data is in the expected direction, since even the extremes on the wave-record curve cannot lie outside the envelope curve. The magnitude of the deviation would be expected to vary from one wave record to another in accordance with the relative bandwidth of the spectrum. Present indications are that a rough estimate of this spectral characteristic will provide enough information to eliminate a large part of the average discrepancy between the points and the line in Figure 5.

Based on the present data, the empirical relation between mean wave height and ordinate dispersion becomes

$$\overline{H} = 2(0.85)\ \sigma_o\ (\pi/2)^{\frac{1}{2}} \qquad \ldots (6)$$

Substituting this empirical relation for the relation (2), we may replace the straight lines of Figure 1 by those in Figure 6, which shows the resulting semi-empirical set of wave-height distributions for various ordinate dispersions. Given the ordinate dispersion σ_o, the corresponding line on Figure 6 represents the expected distribution of wave-heights given by the Rayleigh distribution whose parameters are obtained from the line fitted to the data in Figure 5.

As regards the scatter of the points, the significant fact which Figure 5 illustrates is that the sample-length and spectral bandwidth of the pressure-wave records analyzed were of suitable magnitudes to create a fairly close relationship between the measured estimates of mean wave height and ordinate dispersion. In other words a knowledge of the measured ordinate r.m.s. value (or the total spectral power) for these wave-record samples is essentially equivalent to a knowledge of the measured mean wave height, within 5% over two-thirds of the time. For longer wave records it may be expected that the percent of scatter in Figure 5 will be reduced. As has been seen, the mean wave height essentially determines the distribution of all wave heights. What the present study shows is that sampling fluctuations do not seriously alter the effective validity of the theoretical relationship when 20-minute pressure records are used.

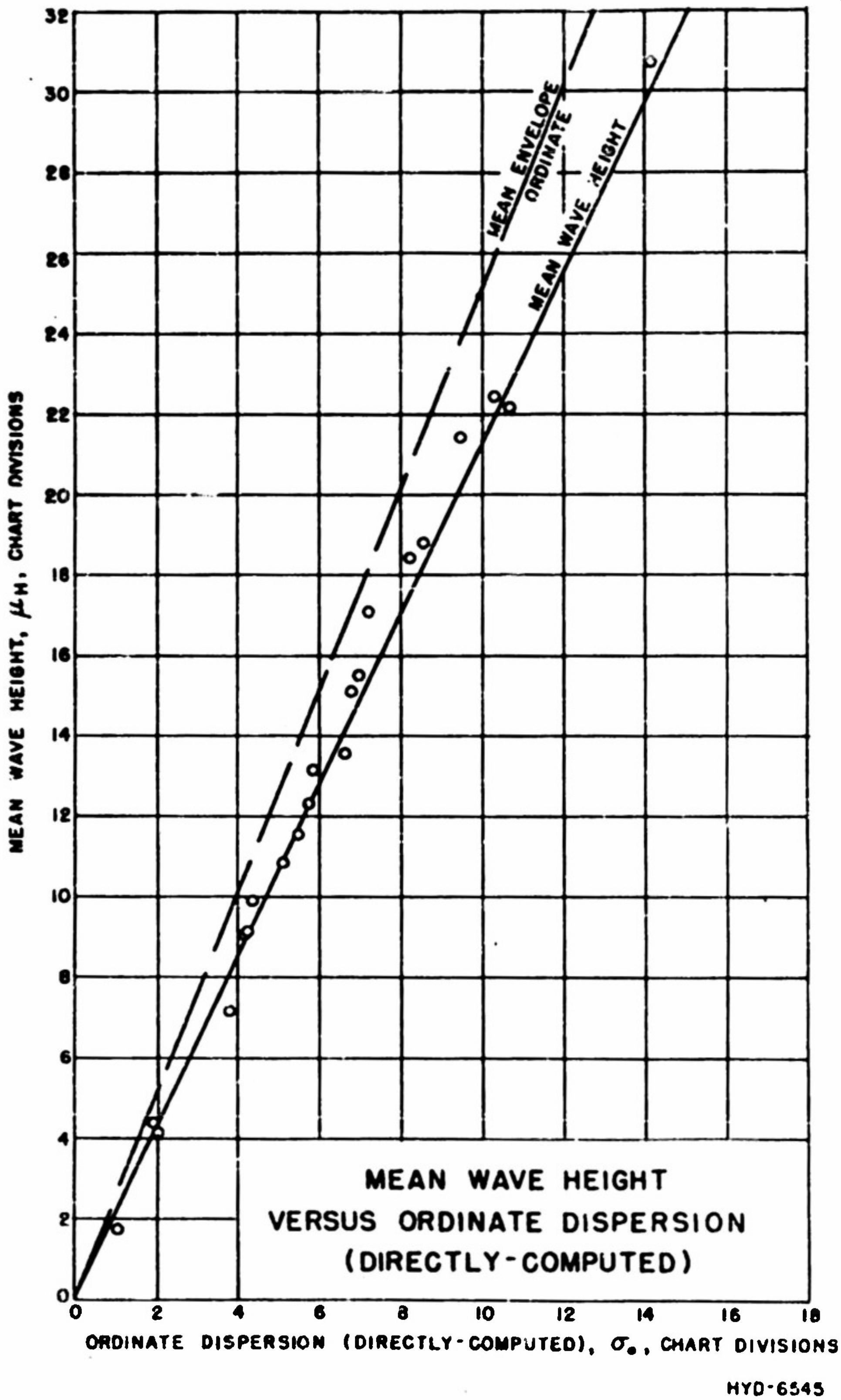
The most important implication of this fact is that for such wave records, a wave-by-wave analysis for wave heights, a procedure relatively difficult to mechanize, may be replaced by an ordinate analysis for σ_o. Since the ordinate dispersion σ_o can be computed as an r.m.s. value, the latter task may be accomplished in principle by means of a direct-reading electrical meter provided with a sufficiently long time-constant.

Under some circumstances the measurement of $\sigma_o = \left\{ \overline{[f(t)]^2} \right\}^{\frac{1}{2}}$ may be less convenient than that of $a_o = \overline{|f(t)|}$, the mean absolute deviation of the ordinates to the wave record (measured from the mean ordinate). Electrical measurement of a_o is possible in principle by means of a rectifying circuit. Since for a Gaussian distribution we have

$$\sigma_o = a_o\ (\pi/2)^{\frac{1}{2}}, \qquad \ldots (7)$$

the mean envelope height is given by the theoretical relation

$$\mu_R = a_o\ (\pi/2), \qquad \ldots (8)$$


MEAN ENVELOPE ORDINATE
MEAN WAVE HEIGHT
MEAN WAVE HEIGHT
VERSUS ORDINATE DISPERSION
(DIRECTLY-COMPUTED)
MEAN WAVE HEIGHT, μ_H, CHART DIVISIONS
ORDINATE DISPERSION (DIRECTLY-COMPUTED), σ_e, CHART DIVISIONS
0
2
4
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18
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FIGURE 5

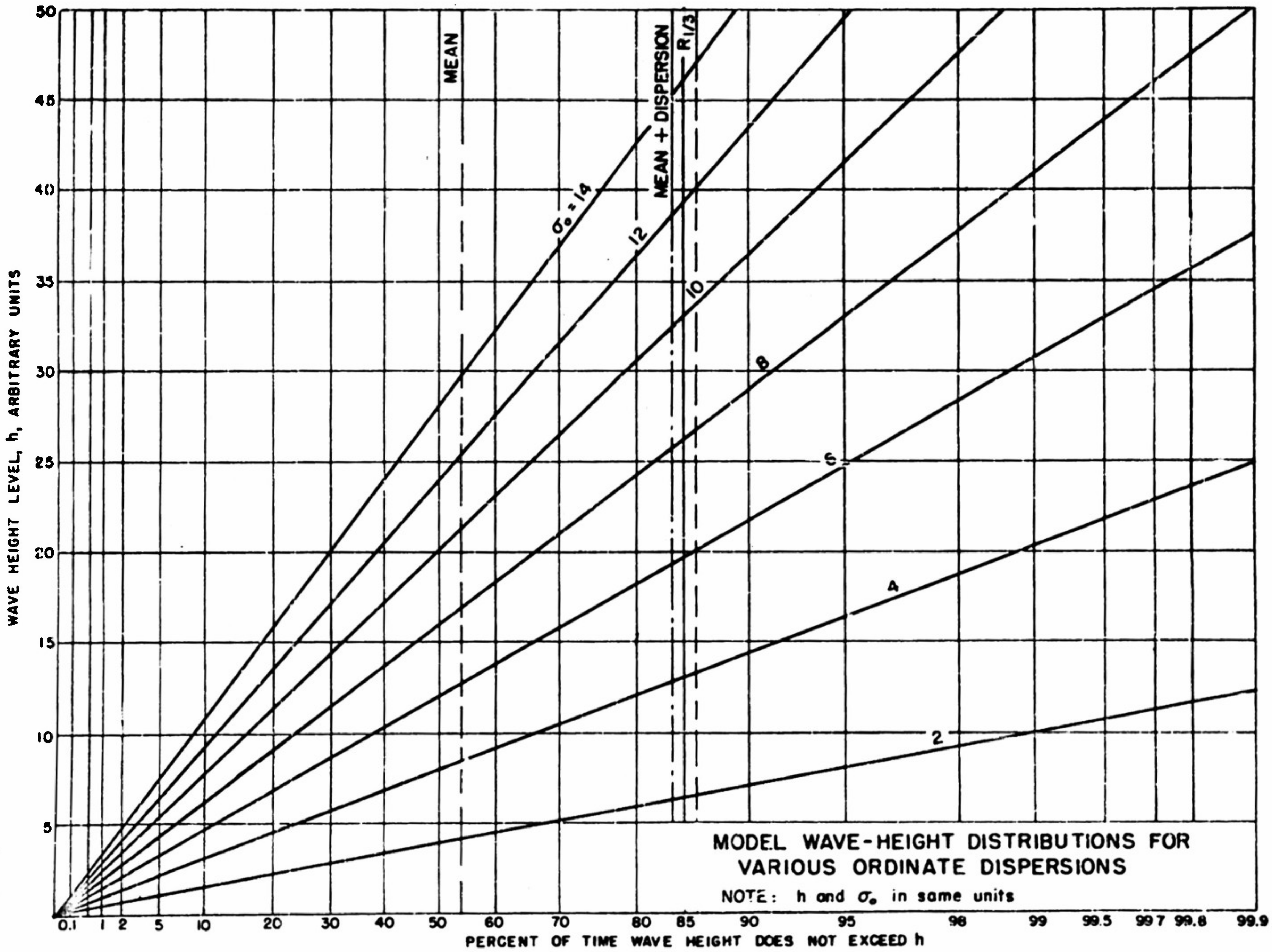

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FIGURE 6

and the mean wave height by the empirical relation

$$\mu_H = 2(0.85)\, a_o\, (\pi/2) \qquad \ldots (9)$$

The sampling stability of the ratio a_o/σ_o, whose expected value is $(2/\pi)^{\frac{1}{2}} = 0.80$, given by (7), is indicated in the results published by Seiwell[18] for wave records slightly less than six minutes in length. The values were reported to be within the limits 0.78 - 0.84, a range in which a reduction could be expected if twenty- or thirty-minute wave records were taken.

Obviously, the relations developed here between various distribution parameters might be made the basis of a number of graphs similar to Figures 1 and 6, showing the distribution of either the theoretical envelope R(t) or the wave heights (H_k) for various values of any one of the following parameters: Ordinate dispersion (σ_o), ordinate mean absolute deviation (a_o); mean wave height (μ_H), wave-height dispersion (σ_H), mean highest one-third wave height ($H_{1/3}$); envelope mean (μ_R), envelope dispersion (σ_R) or envelope mean highest one-third ($R_{1/3}$). Such graphs would be of varying interest, depending upon which parameter was considered as being the basic measured one.

Since the most fundamental parameter is the ordinate dispersion σ_o (or the total spectral power σ_o^2), we summarize in Table II the relations between the parameters by giving the ratio of each to σ_o. The ratio of any other pair of parameters may be found from these ratios by performing a single division. The ratios in Table II may be used to replace the parameters marked on the straight lines in Figures 1 and 6 by any other parameters, e.g. μ_H, $H_{1/3}$, or a_o.

Table II

Ratios of Various Distribution Parameters to Ordinate Dispersion

parameter	a_o	μ_R	σ_R	$R_{1/3}$	μ_H	σ_H	$H_{1/3}$
(parameter)/σ_o	$(2/\pi)^{\frac{1}{2}}$	$(\pi/2)^{\frac{1}{2}}$	$[(4-\pi)/2]^{\frac{1}{2}}$	(cf. pg. 4)	$1.7(\pi/2)^{\frac{1}{2}}$	$1.7\left(\frac{4-\pi}{2}\right)^{\frac{1}{2}}$	$1.7\left(\frac{R_{1/3}}{\sigma_o}\right)$
	0.798	1.253	0.655	2.002	2.130	1.114	3.403

In making use of these ratios it must of course be remembered that they describe theoretical probability distributions, and can only be expected to approximate closely the computed statistics of an empirical frequency distribution if the latter are based on a sufficiently long sequence of data, i.e., a sufficiently long wave record. The magnitude of the sampling fluctuations in certain statistics for twenty-minute wave records has been indicated by results in previous reports[12],[13].

6. Simplified Estimation of Pressure Ordinate Dispersion

The near-rectilinearity of the plots in Figure 3 makes possible for each wave record a convenient graphical estimate of the mean ordinate μ_o and the root-mean-square (or standard) deviation σ_o from the mean ordinate. The first is read directly off the graph as the intercept at the 50%-point on a visually-fitted line. The second is taken as one-half the difference between the intercepts at the 84.1% and 15.9% points, a measure of the slope of the line.

The degree of association between the graphical estimate of σ_o and the mean wave height is shown in Figure 7. The graphical estimate of σ_o obtained from the visually-fitted line on Gaussian probability paper appears to be equivalent, if multiplied by a constant, to the mean pressure-wave amplitude within about 12% in nearly all cases. In fact this graphical method, carried out before the direct-computation method was employed, appears to be almost as good as the latter, as can be seen by a comparison of Figures 5 and 7.

It is evident that for wave systems having ordinate distributions sufficiently close to the Gaussian type, the complete sets of points shown on Figure 3 should not be necessary for obtaining reasonably good estimates of the slopes σ_o. For example, it might be possible to use two counters set at ordinate levels y_o and y_1, chosen in accordance with the expected ordinate dispersion, i.e. the average wave height to be encountered. A two-counter analyzer might be suitable for field installations in which either a stamped or a photographic record of counter readings made at regular intervals would be obtained as the only data.

To estimate the ordinate dispersion from two counter readings, one would, as in Figure 3, plot the two counter levels y_o, y_1 against the two relative counter readings, $P(y_o)$, $P(y_1)$, obtained by dividing the actual counter readings by the total count (which might conveniently be made 12,500 or 20,000 0.1-second time units). These two points determine a unique line on which a certain increment measured on the vertical scale is cut off by the 50% and the 84.1% points. The quantity σ_o', the desired estimate of σ_o, is this vertical increment, which may also be computed numerically by expressing the slope of the line analytically as a function of y_o, y_1, $P(y_o)$, and $P(y_1)$, and making use of a table of the Gaussian probability function or the error function.

The latter procedure was adopted for computing the estimates σ_o', which are plotted versus the directly-computed σ_o in Figure 8. To obtain these estimates for each wave record, y_o and y_1 were selected, as described below, from the ten counter levels used in analyzing the data. It will be seen that the estimate of ordinate dispersion obtained from the two-counter data is relatively close to that obtained using all ten counters. Only once among these wave records does it differ by more than ten percent, and over four-fifths of the time the relative difference was less than five percent.

The estimation procedure may be further simplified by eliminating one of the two counters if y_o can be taken as the mean ordinate or the ordinate

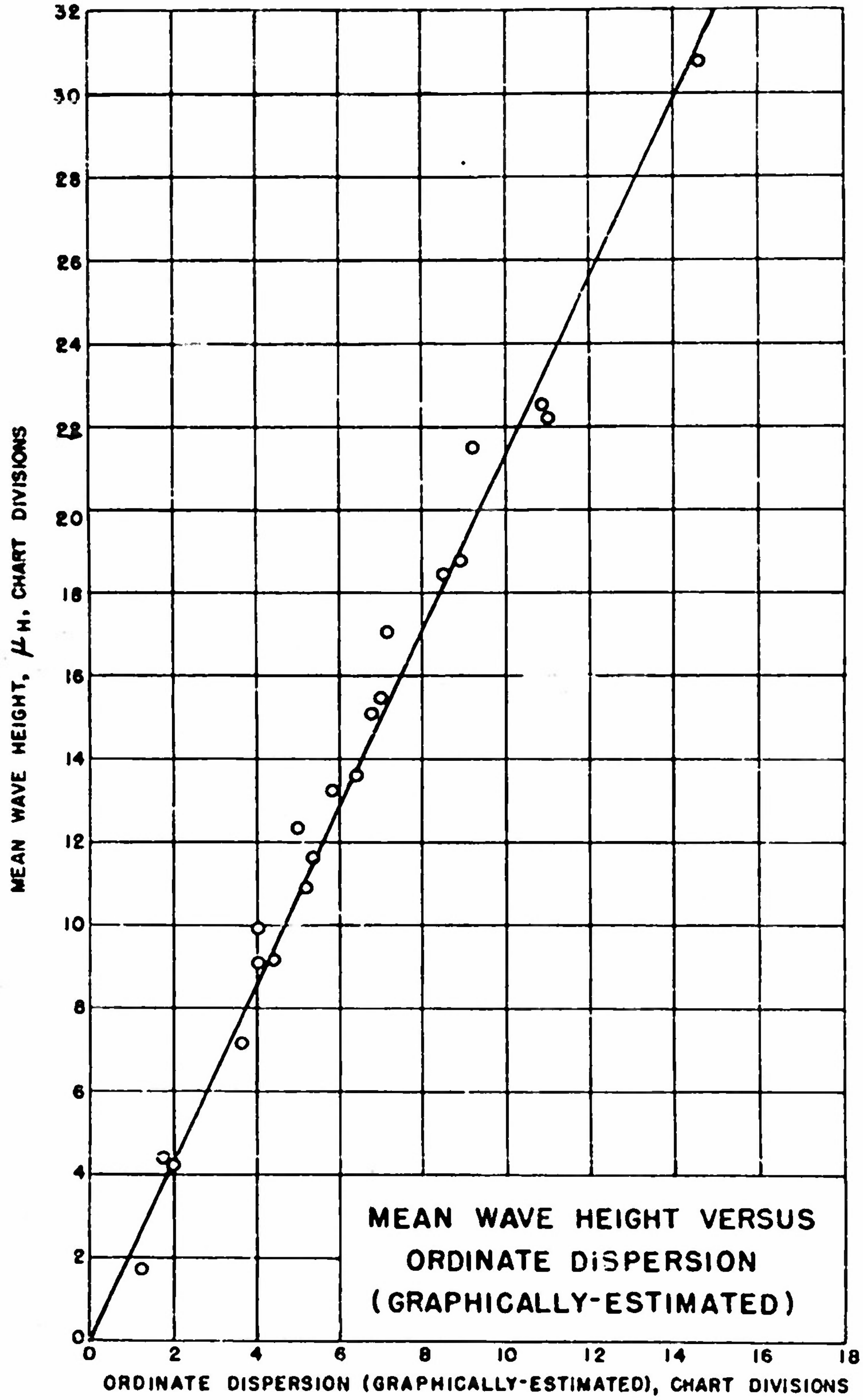

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FIGURE 7

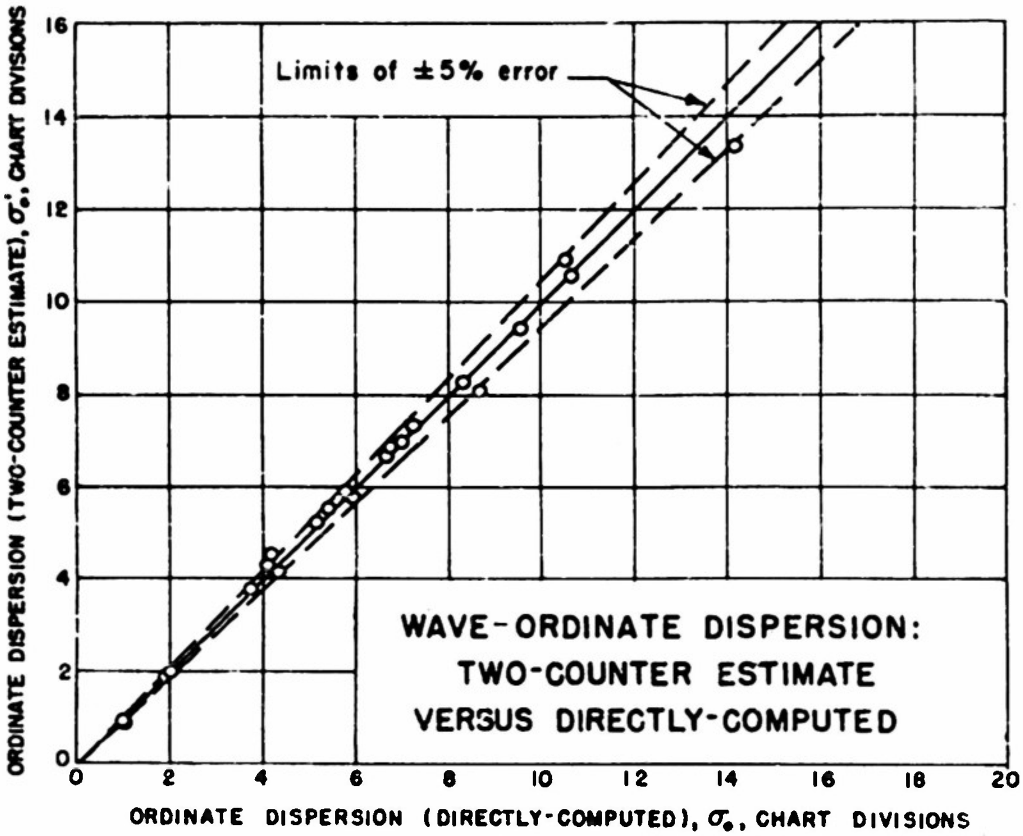

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FIGURE 5

corresponding to the zero differential-pressure level, whose location may be pre-determined for a given instrument installation. If this is done, then $P(y_o)$ will be 0.50, and one of the two plotted points is fixed. The location of the other point (y_1, $P(y_1)$) determines the estimate, σ_o", which will be obtained.

Figure 9 shows, for various counter levels y_1, the curves giving the estimated value σ_o" as a function of the single counter reading $P(y_1)$ expressed in percent of the total count. For example if the counter is set at 4 feet and reads 94.3%, the estimate σ_o" is 2.5 feet. From this value it may also be inferred that, e.g., $H_{1/3}$ is 2(0.85)(4.0) = 6.8 feet, etc. As explained earlier, use may be made of the ratio between σ_o and any other parameter of interest to make the vertical scale in Figure 9 direct-reading for that parameter - e.g. μ_R, μ_H, $H_{1/3}$, etc.

While the average values of the estimates σ_o" are indicated by Figure 9, the statistical fluctuation of the estimates based on the one-counter analysis of a twenty-minute record may be visualized from Figure 3. The two dashed lines on the plots in Figure 3 have been drawn to correspond to the upper and lower 10% limits for the error in predicting σ_o from individual plotted points when the mean ordinate μ_o is known. The estimate of σ_o obtained in this way from any <u>one</u> point is within ±10% of the value computed directly from all ten points if and only if that point lies in the two narrow sectors between the two dotted lines. It can be seen that percentage errors are apt to be high for single points which are either too near the center of the distribution or else too far out on its extremes. There are indications that the most reliable estimates of σ_o are obtained when the ordinates y_1 and y_o are chosen symmetrically about the mean at a distance of about 1.5 σ_o. These positions in the Gaussian distribution of ordinates correspond to counter settings at the 93% and the 7% points, respectively. The dotted vertical line in Figure 9 allows this setting of y_1 to be read off directly for any given ordinate dispersion σ_o. The counter readings used for the estimates σ_c' shown in Figure 8 were chosen to be those most nearly at these vertical levels.

Although σ_o is initially unknown, a relatively small amount of preliminary observation would allow the counter to be set for further observation near the optimum level +1.5 σ_o, or the 93% point, in terms of the counter reading. For a permanent installation y_1 could be automatically adjusted at intervals, if necessary. For example, the counter level may be shifted every half-hour to +1.5 times the previous estimate of σ_o, thus keeping approximately in step with the slow changes in the total wave-spectrum power. If such shifts in y_1 were made from time to time to take account of changing wave conditions, it would be necessary, of course, to record each new level y_1.

In the absence of self-adjusting counter levels, one or two fixed levels could be chosen as often as feasible in accordance with the expected values of σ_o for the wave conditions of interest most likely to occur over an observation interval of, perhaps, days or weeks. In this case a knowledge of the <u>wave climate</u> for the place and season of installation could be used

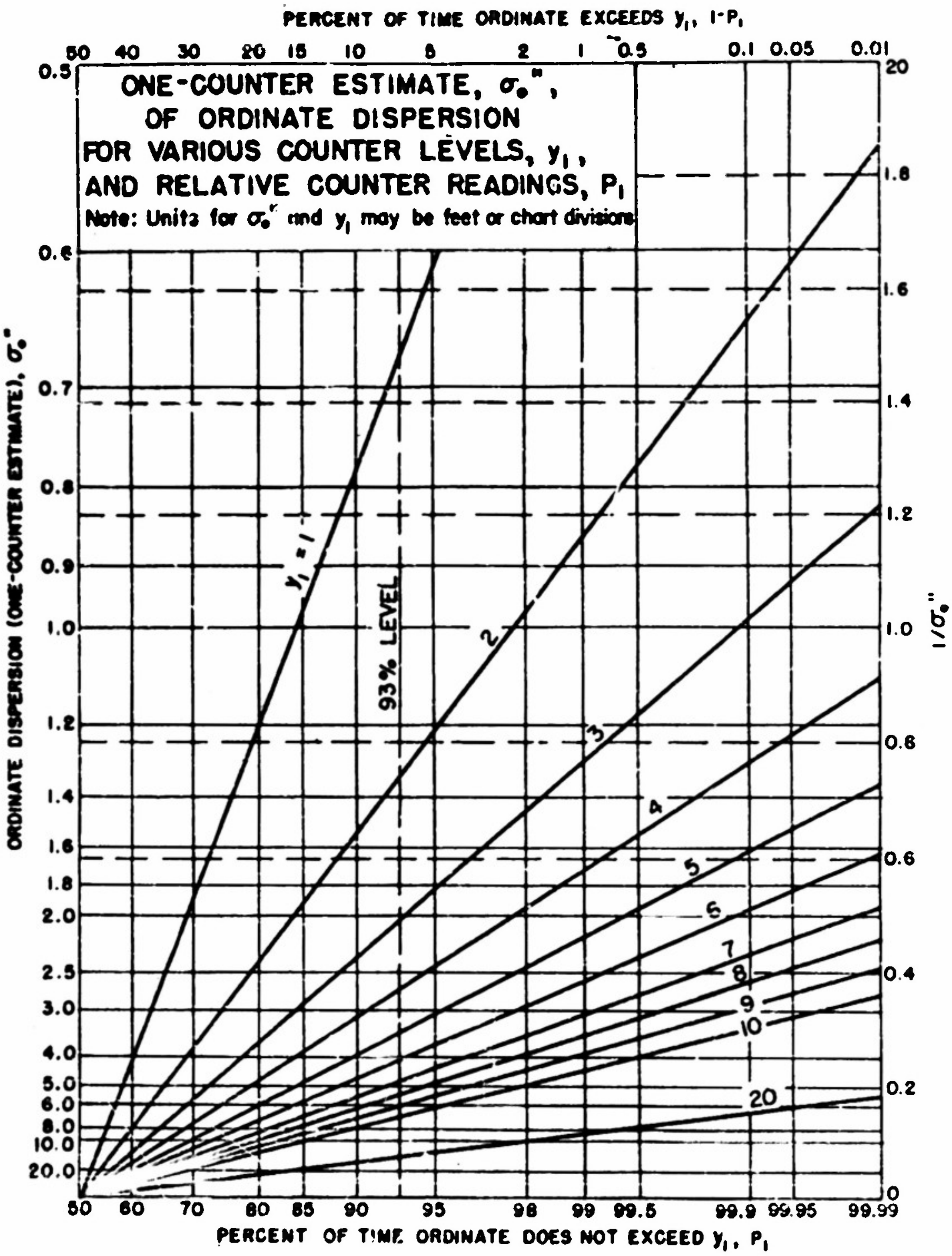

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FIGURE 9

as a basis for selecting the fixed ordinate levels. This type of wave-climate information should be available after sufficient data taken in this way at the installation site has accumulated. In making use of such information a compromise would be found necessary between (1) the loss of accuracy for high wave records when the counter level is chosen too low, and (2) the loss of nearly all information from low wave records when the counter level is chosen too high.

As an example of the application of the fixed-level procedure to a given wave-meter installation, the following hypothetical situation may be imagined. Suppose that the twenty-one wave records E, F, ..., Y had been obtained at a single location (instead of four) over a single season of the year (instead of all seasons). If this data were considered as historical wave-climate data for such an installation, the anticipated average value of σ_0 would be approximately 6.0 chart divisions above the mean. The low wave records show, however, that unless the counter level is made as low as about 4.0 units above the mean, no information, except an upper limit to the ordinate distribution, would be obtained in some cases.

A rough, probably conservative, test of this one-counter, fixed-level method of estimating σ_0 was conducted by choosing the level y_1 to be the level of the counter which, in the original analysis of the present data, was set nearest to 4.0 chart divisions above the mean. As a result, for either (1) low waves such as in Wave Record "U" or (2) high waves such as in Wave Record "V", the point chosen is far from the 93% level, and the estimated slope, σ_0". is subject to relatively large errors. These will result from either (1) division by a small number or (2) multiplication by a relatively unstable observed value far out in the tail of the ordinate distribution.

From the counter levels shown in Figure 3 it may be seen that most of the points, the distance of whose ordinate above the mean is nearest 4.0 chart divisions, would fall within a ±10% error band. The values of the estimate, σ_0", were computed analytically and plotted in Figure 10 against the values of σ_0 already computed by using the data from all ten counters. The two outer lines in the figure indicate the 5% error limits.

The advantage of an approximate knowledge of the 93% ordinate level is shown by a comparison of the estimates plotted in Figure 10 with those plotted in Figure 11. The latter estimates σ_0''' were based on a knowledge of the mean-ordinate level and were obtained from the reading of the counter nearest the 93% level (i.e. one of the counters used in computing the two-counter estimate of the ordinate dispersion).

Figures 8, 10 and 11 indicate that satisfactory estimation of the r.m.s. value (and hence mean wave height, etc.) of a wave record may be achieved by the use of only one or two ordinate counters. Such a routine estimation should provide useful information about wave records, if used over a suitable length of time in a number of installations. If a sequence of estimates σ_0" were made, say, at one-half hour intervals, the results would be to some extent self-checking, in that values differeing markedly from the general trend could be interpreted as having been overly influenced by random factors. Thus significant trends in σ_0 reflecting 100-200% changes in wave energy would not be readily obscured by an occasional random 10% change observed in σ_0".

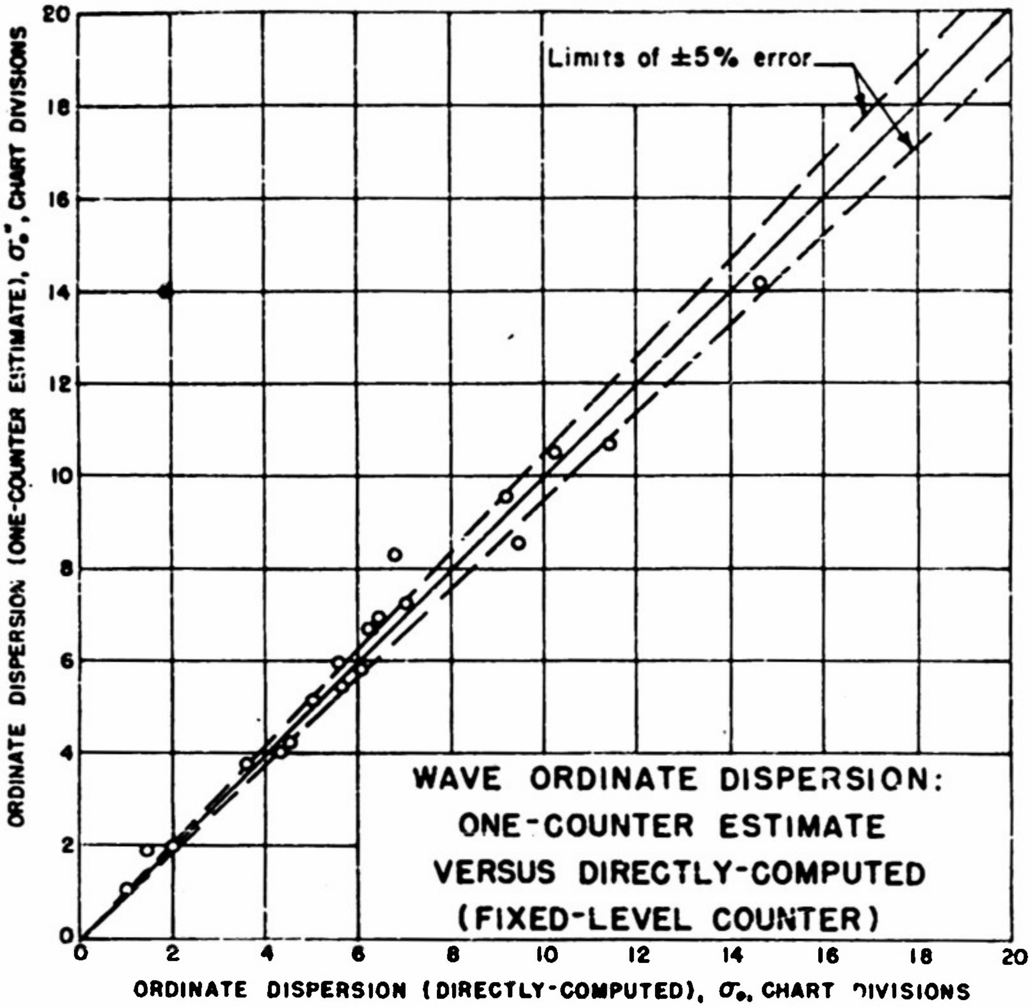

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FIGURE 10

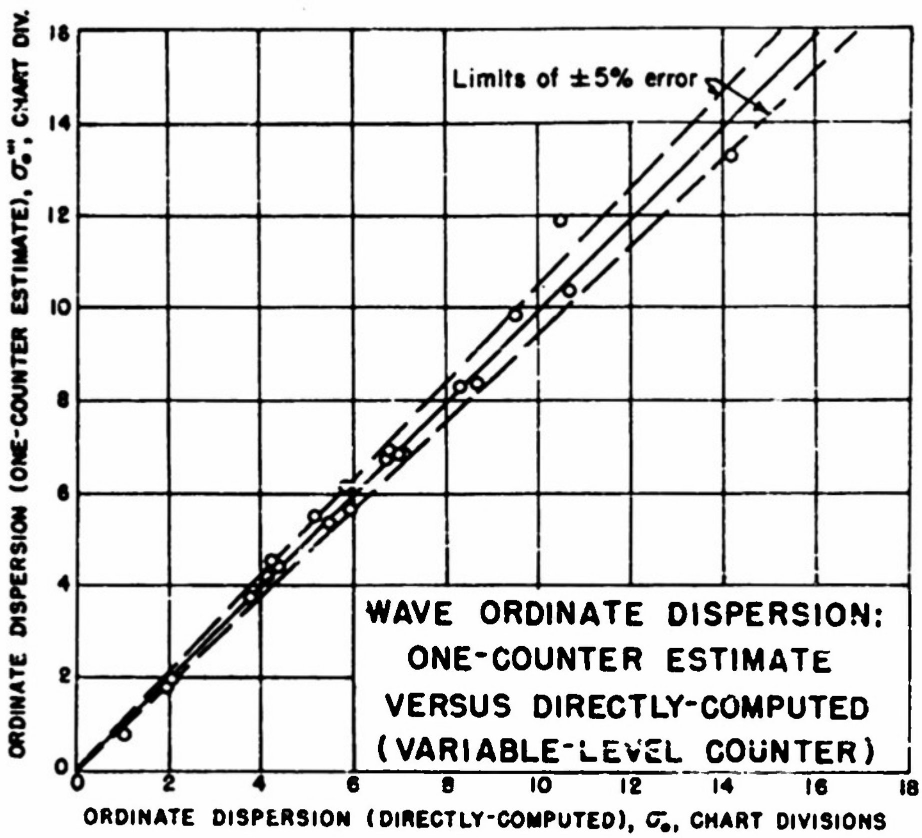

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FIGURE 11

CONCLUSIONS

Ocean-wave record ordinates are found to have approximately the following properties of a narrow-spectrum Gaussian random process: (1) Pressure wave-record ordinates follow closely the Gaussian distribution. (2) Estimated surface wave heights follow closely the Rayleigh distribution. (3) Estimated surface wave-height distribution parameters are related to one another in the same way as are those of the Rayleigh distribution, all being closely proportional to the pressure ordinate dispersion σ_0. (4) Pressure wave-height mean and pressure wave-ordinate dispersion are closely proportional.

For pressure wave records, wave-height mean is found to be approximately 1.70 $(\pi/2)^{1/2}$ times wave-ordinate dispersion. Graphical estimation of wave-ordinate dispersion is found to be a reasonably accurate method of estimating wave-height mean for twenty-minute pressure records of ocean swell. A fixed, suitably-situated, single point on the ordinate distribution curve is found to yield a reasonably accurate estimate of ordinate dispersion if the "still-water level" is known, while the use of one or two variable points gives improved results.

A systematic program of ocean-wave measurement utilizing these simplified forms of the Wave-Ordinate Distribution Analyzer may be expected to provide useful continuous wave-height information in immediately-available form.

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